Tidal gravitational effects in a satellite.

Philippe Tourrenc, Marie-Christine Angonin-Willaime, Xavier Ovido

Tidal gravitational effects in a satellite.

Ph. Tourrenc, M-C. Angonin, and X. Ovido
Université P. et M. Curie
ERGA, case 142
4, place Jussieu
F-75252 Paris CEDEX 05, France
(Dated: July 23, 2004)

Atomic wave interferometers are tied to a telescope pointing towards a faraway star in a nearly free falling satellite. Such a device is sensitive to the acceleration and the rotation relatively to the local inertial frame and to the tidal gravitational effects too.

We calculate the rotation of the telescope due to the aberration and the deflection of the light in the gravitational field of a central mass (the Earth and Jupiter). Within the framework of a general parametrized description of the problem, we discuss the contributions which must be taken into account in order to observe the Lense-Thirring effect.

Using a geometrical model, we consider some perturbations to the idealized device and we calculate the corresponding effect on the periodic components of the signal.

Some improvements in the knowledge of the gravitational field are still necessary as well as an increase of the experimental capabilities ; however our conclusions support a reasonable optimism for the future.

Finally we put forward the necessity of a more complete, realistic and powerful model in order to obtain a definitive conclusion on the feasibility of the experiment as far as the observation of the Lense-Thirring effect is involved.

I. INTRODUCTION

Clocks, accelerometers and gyroscopes based on cold atom interferometry are already among the best which have been constructed until now and further improvements are still expected. With the increase of the experimental capabilities it becomes necessary to consider more and more small effects in order to account for the signal, therefore (relativistic) gravitation has to be considered in any highly sensitive experiments, no matter what they are designed for.

The performances of laser cooled atomic devices is limited on Earth by gravity. Further improvements demand now that new experiments take place in free falling (or nearly free falling) satellites. A laser cooled atomic clock, named PHARAO, will be a part of ACES (Atomic Clock Ensemble in Space), an ESA mission on the ISS. Various other experimental possibilities involving "Hyper-precision cold atom interferometry in space" are presently considered. They might result in a project (called "Hyper") in the future [1].

The aim of the present paper is to hold the bookkeeping of the various gravito-inertial effects in a nearly free falling satellite. For this purpose we consider the most ambitious goal which has been considered for Hyper *i.e.* the measurement of the Lense-Thirring effect.

The Lense-Thirring effect is a local rotation of a gyroscope relatively to a telescope pointing towards a far away star. It is a relativistic consequence of the diurnal rotation of the Earth which "drags the inertial frames" in its neighborhood.

The angular velocity of the telescope relative to the gyroscopes depends on the position. Therefore, in a satellite, it is a function of the time. In Hyper, the angular velocity is measured by atomic-wave-gyroscopes and its time dependence is analyzed [9]. The consequence is that the device is sensitive to the variation of the gravitation in the satellite and no to the gravitation itself. We do not believe that it is easy to achieve the required stabilization of the gravitational field due to the local masses but it is not impossible in principle. For this reason we will study only the tidal field of far away masses whose effect cannot be removed at all.

The parameter which plays a role in the calculation of the Lense-Thirring effect is the angular momentum of the central mass. It is much bigger for Jupiter than the Earth. Therefore we will discuss both cases, without any consideration on the cost of the corresponding missions.

In the sequel the greek indices run from 0 to 3 and the Latin indices from 1 to 3. We use the summation rule of repeated indices (one up and one down).

The Minkowski tensor is $\eta_{\alpha\beta} = diag\,[1,-1,-1,-1]$; its inverse is $\eta^{\alpha\beta}$.

We use geometrical units where $c = G = 1$.

II. THE LOCAL EXPERIMENT IN A SATELLITE

In the satellite, the experimental set-up consists in a telescope pointing towards a far away star in the $\overrightarrow{u}_{(1)}$ direction and two orthogonal atomic Sagnac units in the planes $\left[\overrightarrow{u}_{(3)}, \overrightarrow{u}_{(1)}\right]$ and $\left[\overrightarrow{u}_{(2)}, \overrightarrow{u}_{(1)}\right]$ of fig. 1.

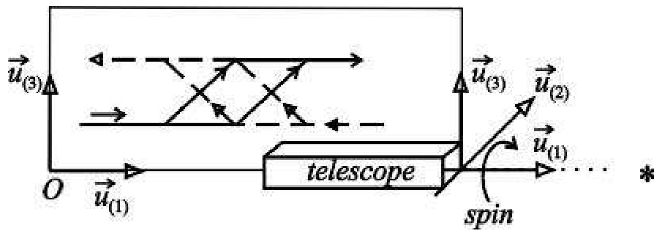

FIG. 1: The experimental setup.

A. The atomic Sagnac unit

An atomic Sagnac unit (ASU) is made of two counter-propagating atom interferometers which discriminate between rotation and acceleration (see figure 2-a).

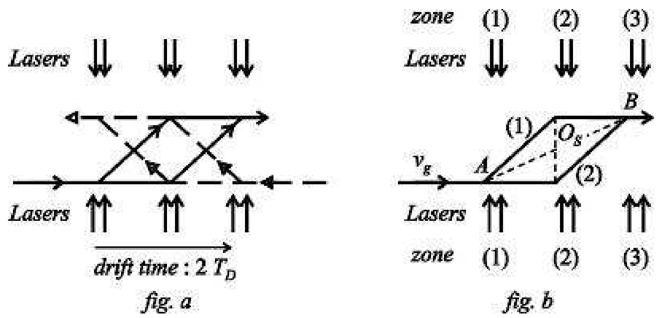

FIG. 2: An atomic Sagnac unit (ASU).

Each interferometer is a so-called Ramsey-Bordé interferometer with a Mach-Zehnder geometry (figure 2-b). The atomic beam from a magneto-optical trap interacts three times with a laser field. In the first interaction zone the atomic beam is split coherently, by a Raman effect, into two beams which are redirected and recombined in the second and the third interaction zone.

The mass of the atom depends on its internal state, therefore it is not a constant along the different paths. However, the change of the mass is very small; it leads to negligible corrections on the main effects which is already very small. In the case of the cesium, the mass is $m = 133 \times 1.66 \times 10^{-27} = 2.2 \times 10^{-25}$kg and the wave length of the lasers is $\lambda = 850$nm. The momentum transferred to the atom during the interaction is $\dfrac{4\pi\hbar}{\lambda}$. The recoil of the atom results in a Sagnac loop which permits to measure the angular velocity of the set-up relatively to a local inertial frame. The device is also sensitive to the accelerations.

In an ideal set-up the two interferometers are identical coplanar parallelograms with their center O_S and O'_S at the same point but many perturbations have to be considered. The geometry of the device is actually determined by the interaction between the initial atomic beam and the lasers ; Therefore a full treatment of the atom-laser interaction in a gravitational field is obviously necessary to study the response of the Atomic Sagnac Unit (ASU). However the geometrical model is useful to give a physical intuition of the phenomena. In this context we assume that the two interferometers remain idealized identical parallelograms but that O_S and O'_S are no longer at the same point : This is the only perturbation that we consider here. It is sufficient to take the flavor of the gravitational perturbations which have to be taken into account and, more generally, of the difficulty inherent to such an experiment.

B. The phase difference

Let us assume that the fundamental element is known in some coordinates comoving with the experimental set-up :

$$\begin{aligned} ds^2 &= \left(1+K_{(0)(0)}\right)dT^2+2K_{(0)(k)}dTdX^{(k)} \\ &\quad +\left(\eta_{(k)(j)}+K_{(k)(j)}\right)dX^{(k)}dX^{(j)} \end{aligned} \tag{1}$$

In order to calculate up to first order the gravitational perturbation of the phase due to $K_{(\alpha)(\beta)}$, we use a method which we summarize now [2].

First we calculate the quantity Ψ :

$$\Psi = K_{(0)(0)}+2K_{(0)(k)}v_g^{(k)}+K_{(k)(j)}v_g^{(k)}v_g^{(j)} \tag{2}$$

where $v_g^{(k)}$ is the velocity of the atoms (*i.e.* the unperturbed group velocity).

The quantity Ψ is a function of the time and the position of the atom.

Then we consider an atom which arrives at time t at point B of figure 2-b . Now the position is a function of the time t' only because t is considered as a given quantity. The function Ψ is a function of the time, t', only. The phase difference is

$$\delta\varphi = \frac{\omega}{2}\int_{t-2T_D/(2)}^{t}\Psi\left(t'\right)dt' - \frac{\omega}{2}\int_{t-2T_D/(1)}^{t}\Psi\left(t'\right)dt' \tag{3}$$

The integrals are performed along path (2) and (1) of the interferometer (figure 2-b). The "angular frequency" ω is defined as $\dfrac{m\,c^2}{\hbar}$.

C. The local metric

In order to calculate $\delta\varphi$, we must know the local metric $G_{(\alpha)(\beta)} = \eta_{(\alpha)(\beta)}+K_{(\alpha)(\beta)}$.

We choose an origin, O, in the satellite and at point O a tetrad $\left\{u^\alpha_{(0)},u^\alpha_{(1)},u^\alpha_{(2)},u^\alpha_{(3)}\right\}$ where $u^\alpha_{(0)}$ is the 4-velocity of point O and where the three vectors $\left\{u^\alpha_{(k)}\right\} = \left\{0, \overrightarrow{u}_{(k)}\right\}$ are represented on the figure 1. The vectors of the tetrad are orthogonal : $u^\alpha_{(\mu)} u_{\alpha(\sigma)} = \eta_{(\mu)(\sigma)}$. The coordinate indices, α, β, σ, etc, are lowered or raised by the means of the metric tensor, $g_{\alpha\beta}$ or $g^{\alpha\beta}$; the Minkowski indices are raised or lowered with the Minkowski metric or its inverse, $\eta_{\alpha\beta}$ or $\eta^{\alpha\beta}$. An Einstein indice, α, can be changed into a Minkowski indice (ρ), by the means of the tetrad and vice versa : $u^\alpha_{(\rho)}\,(\ )_\alpha = (\ )_{(\rho)}$ and $u^\alpha_{(\rho)}\,(\ )^{(\rho)} = (\ )^\alpha$.

The tetrad is the natural basis at point O of comoving coordinates $X^{(\alpha)}$. The proper time at the origin is $X^{(0)}$. The space coordinates are the $X^{(k)}$.

Once the origin and the tetrad are chosen, the metric at point M and time t is expanded relatively to the space coordinates of M [3].

$$ds^2 = G_{(\alpha)(\beta)}\, dX^{(\alpha)} dX^{(\beta)} \quad \text{with} \tag{4}$$

$$\begin{aligned}
G_{(0)(0)} &= 1 + 2\,\vec{a}\cdot\vec{X} + \left(\vec{a}\cdot\vec{X}\right)^2 - \left(\vec{\Omega}\times\vec{X}\right)^2 - R_{(0)(k)(0)(j)}\, X^{(k)} X^{(j)} \\
&\quad -\frac{1}{3} R_{(0)(k)(0)(j),(\ell)} X^{(k)} X^{(j)} X^{(\ell)} + \ldots \qquad (5)\\
G_{(0)(m)} &= \Omega_{(m)(k)}\, X^{(k)} - \frac{2}{3} R_{(0)(k)(m)(j)}\, X^{(k)} X^{(j)} \\
&\quad -\frac{1}{4} R_{(0)(k)(m)(j),(\ell)} X^{(k)} X^{(j)} X^{(\ell)} + .. \\
G_{(n)(m)} &= \eta_{(n)(m)} - \frac{1}{3} R_{(n)(k)(m)(j)} X^{(k)} X^{(j)} \\
&\quad -\frac{1}{6} R_{(n)(k)(m)(j),(\ell)} X^{(k)} X^{(j)} X^{(\ell)} + \ldots
\end{aligned}$$

where we have used vector notations *i.e.* $\overrightarrow{a}$ for $\left\{a^{(\ell)}\right\}$, $\overrightarrow{a}\cdot\overrightarrow{X}$ for $\sum a^{(\ell)}\, X^{(\ell)}$, etc. Every quantity, except the space coordinates $X^{(\ell)}$, are calculated at point O. Thus they are functions of the time $T = X^{(0)}$.

$R_{(\alpha)(\beta)(\sigma)(\mu)}$ is the Riemann tensor obtained from $R_{\alpha\beta\sigma\mu}$ at point O :

$$R_{\alpha\beta\sigma\mu} = \Gamma_{\alpha-\beta\mu,\sigma} - \Gamma_{\alpha-\beta\sigma,\mu} + \Gamma^\varepsilon_{\beta\sigma}\Gamma_{\varepsilon-\alpha\mu} - \Gamma^\varepsilon_{\beta\mu}\Gamma_{\varepsilon-\alpha\sigma} \tag{6}$$

where $\Gamma_{\alpha-\beta\mu,\sigma}$ is the Christoffel symbol.

$\Omega_{(j)(k)}$ is the antisymmetric quantity

$$\begin{aligned}
\Omega_{(j)(k)} &= \frac{1}{2}\left(g_{(0)(j),(k)} - g_{(0)(k),(j)}\right)_O \qquad (7)\\
&\quad + \frac{1}{2}\left(\left(u^\beta_{(j)}\, \frac{du^\alpha_{(k)}}{ds} - \frac{du^\beta_{(j)}}{ds}\, u^\alpha_{(k)}\right) g_{\alpha\beta}\right)_O
\end{aligned}$$

Due to the antisymmetry of $\Omega_{(m)(k)}$, the quantity $\Omega_{(m)(k)}\, X^{(k)} dX^{(m)}$ which is present in the expression of ds^2 can be written as $\Omega_{(m)(k)}\, X^{(k)} dX^{(m)} = \left(\overrightarrow{\Omega}_0 \wedge \overrightarrow{X}\right) \cdot d\overrightarrow{X}$. The space vector $\overrightarrow{\Omega}_0$ is the physical angular velocity. It is measured by gyroscopes tied to the three space orthonormal vectors $u^{\alpha}_{(k)}$:

The vector $\overrightarrow{a}$ is the physical acceleration which can be measured by an accelerometer comoving with O. It is the spatial projection at point O of the 4-acceleration of point O.

At point O (*i.e.* $\overrightarrow{X} = \overrightarrow{0}$) the time T is the proper time delivered by an ideal clock comoving with O.

III. FROM THE GEOCENTRIC COORDINATES TO THE COMOVING COORDINATES

A. The geocentric coordinates

We define the time coordinate $x^0 = ct$ and the space coordinates x^k. We use the notations $\vec{r} = \left\{x^k\right\} = \{x, y, z\}$ and we define the spherical coordinates $\{r, \theta, \varphi\}$, *i.e.* $x = r\sin\theta\,\cos\varphi$, $y = r\sin\theta\,\sin\varphi$, $z = r\cos\theta$.

We consider a satellite and a point O which is the origin of the local coordinates in the satellite. We assume that the position of O is given by its three space coordinates, $\vec{r} = \{x, y, z\} = \left\{x^k\right\}$, considered as three known functions of the coordinate time, t. Then we define the velocity of point O as $\overrightarrow{v} = \dfrac{d\vec{r}}{dt}$.

The proper time at point O is $s = T = X^{(0)}$. The motion of O can be described as well by the four functions $x^{\alpha} = x^{\alpha}(s)$. The four-velocity is defined as $u^{\alpha} = \dfrac{dx^{\alpha}}{ds}$.

In the sequel we consider the Parametrized Post Newtonian theories [4]. The relevant PPN parameters which appear below are γ and α_1. The parameter γ is the usual parameter connected to the deflection of a light ray by a central mass. The parameter α_1 couples the metric to the speed, $-\vec{w}$, of the preferred frame (if any) relatively to the geocentric frame. In general relativity, $\alpha_1 = 0$ and $\gamma = 1$.

Let us define now several quantities which will be used in the sequel :

- $2M_{\circledS}$ is the Schwarzschild's radius of the central body (*i.e.* the Earth or Jupiter). As we use geometrical units, $M_{\circledS}$ is also its "mass".
- $\vec{J}_{\circledS}$ is the angular momentum of the central body in geometrical units. The relevant quantity which appears below, is $\vec{J} = \dfrac{1+\gamma+\alpha_1/4}{2}\vec{J}_{\circledS}$.

We define $J = \left\|\vec{J}\right\| \simeq \left\|\vec{J}_{\circledS}\right\| = J_{\circledS}$

- $\vec{g} = -2\ \dfrac{\vec{J} \wedge \vec{r}}{r^3} + \dfrac{1}{2}\alpha_1 U\ \vec{w}$ is the definition of $\vec{g}$, where $\vec{w}$ is the velocity of the observer, relative to the preferred frame (if any).
- U is the Newtonian potential

$$U = \frac{M_{\circledS}}{r}\left(1 - J_2\left(\frac{R_{\circledS}}{r}\right)^2 P_2 + \Delta\right) + U_* \tag{8}$$

where $R_{\circledS}$ is the radius of the central body and U_* the potential due to its satellites, the Sun and the planets[10]. In spherical coordinates the Legendre polynomial P_2 reads $P_2 = \dfrac{1}{2}\left(3\cos^2\theta - 1\right)$. The quadrupole coefficient is J_2 and Δ represents the higher harmonics. It depends on the angle φ and on the time t because of the rotation of the central body.

In the non rotating geocentric coordinates the significant fundamental element is

$$ds^2 = (1 - 2U)\,dt^2 + 2g_{0k}\,dx^k dt - (1 + 2\gamma U)\,\delta_{jk}\,dx^j dx^k \tag{9}$$

where $(\vec{g})_k = -(\vec{g})^k = g_{0k}$. In eq. (9), we have dropped post Newtonian corrections which are too small to be considered here.

B. Orders of magnitude

Table 1 below gives the order of magnitude of the various parameters which have been introduced previously.

	$M_{\circledS}$	$J_{\circledS}$	$R_{\circledS}$	J_2	Δ
Earth	4.4mm	145cm^2	6400km	$\sim 10^{-3}$	$\sim 10^{-6}$
Jupiter	1.4m	1700m^2	71300km	$\sim 10^{-2}$	$\lesssim 10^{-3}$

Table 1.

In order to describe the physical situation we introduce four parameters : ξ, ε, η and μ.

First we define the order of magnitude $O_1 = \sqrt{\dfrac{M_{\circledS}}{R_{\circledS}}}$. The quantity $(O_1)^n$ is denoted by O_n.

Then we consider a nearly free falling satellite on a nearly circular orbit of radius $r \sim R_{\circledS}/\xi$. This expression gives the definition of ξ. The velocity of the satellite is of order $v = \xi^{1/2} O_1$ [11].

Now we define $d = R_{\circledS} O_1$ and ε such as $X = \varepsilon d$ where X is the size of the laboratory.

We define η. The velocity of the atoms is $v_g = \eta O_1$.

Finally we assume that the various quantities such as the position of O or the geometry of the experimental set-up is known with a relative accuracy of order of μ.

$O_1 = \sqrt{\dfrac{M_{\circledS}}{R_{\circledS}}}$	relative accuracy : μ
Orbital parameters	set-up parameters
radius $r = \dfrac{R_{\circledS}}{\xi}$ velocity $v = O_1 \xi^{1/2}$ period $T = \dfrac{2\pi}{\xi^{1/2} O_1} \dfrac{R_{\circledS}}{c\,\xi}$	size $X = R_{\circledS} O_1\ \varepsilon \sim 60\text{cm}$ atom velocity $v_g = \eta O_1 \sim 20\text{cms}^{-1}/c$ Drift time $2T_D = X/v_g = R_{\circledS}\ \dfrac{\varepsilon}{\eta} \sim 3\text{s}$

Table 2. : definition of O_1, ξ, ε and η

With $\xi \simeq 0.9$ one finds

	O_1	ε	η	r	v	T
Earth	$2.6\ 10^{-5}$	$3.6\ 10^{-3}$	$2.7\ 10^{-5}$	7000km	$2.9\ 10^{-5}$	5900s
Jupiter	$1.4\ 10^{-4}$	$6.0\ 10^{-5}$	$4.8\ 10^{-6}$	78400km	$1.5\ 10^{-5}$	12300s

Table 3.

C. Comoving non rotating coordinates

We consider the following tetrad, $e^{\alpha}_{\hat{\sigma}}$, comoving with O :

$$
\begin{aligned}
e^0_{\hat{0}} &= u^0 = 1 + \frac{\vec{v}^2}{2} + U\ + O_4, \quad e^k_{\hat{0}} = u^k = \left(1 + \frac{\vec{v}^2}{2} + U\right) v^k + \xi^2 O_4 \\
e^0_{\hat{k}} &= \left(1 + \frac{\vec{v}^2}{2} + U\right) v^k + \gamma U\, v^k - g_{0k} + \xi^2 O_4 \qquad (10) \\
e^j_{\hat{k}} &= \delta^j_k + \frac{1}{2} v^j v^k + \frac{1}{2}\gamma U\, \delta^j_k + \xi^2 O_4
\end{aligned}
$$

The local metric is derived from 4 with the change in the notations $(\alpha) \to \hat{\alpha}$ and $u^{\alpha}_{(\sigma)} \to e^{\alpha}_{\hat{\sigma}}$.

We limit the expansion of the metric at order $\varepsilon^2\, \xi^{3/2} O_6$; therefore we consider only the linear expression of the Riemann tensor (eq. (6)) and we assume that the free fall is under control : $\|\vec{a}\| << \varepsilon O_3 \xi^{3/2} \times \dfrac{c^2}{X}$ (*i.e.* $\|\vec{a}\| << 8\text{ms}^{-2}$ for the Earth, $\|\vec{a}\| << 21\text{ms}^{-2}$ for Jupiter which is not very restrictive) therefore we neglect the term $\left(\vec{a} \cdot \vec{X}\right)^2$ in the metric (4). One finds

$$G_{\hat{0}\hat{0}} = 1 + 2\,\vec{a}\cdot\vec{X} - \hat{U}_{,\hat{k}\hat{\jmath}}\,X^{\hat{k}}X^{\hat{\jmath}} - \frac{1}{3}\hat{U}_{,\hat{k}\hat{\jmath}\hat{\ell}}\,X^{\hat{k}}X^{\hat{\jmath}}X^{\hat{\ell}} + \varepsilon^2\,\xi^3 O_6 \quad (11)$$

$$G_{\hat{0}\hat{m}} = -\left\{\vec{\Omega}_0\wedge\vec{X}\right\}^{\hat{m}} + \varepsilon^2\xi^{5/2}O_5 \text{ and } G_{\hat{n}\hat{m}} = \eta_{\hat{n}\hat{m}} + \varepsilon^2\xi^2 O_4 \quad (12)$$

where $\vec{\Omega}_0$ is given below (see eq. (13)) while the expressions such as $\hat{U}_{,\hat{k}\hat{\jmath}}$ are nothing but $\left(U_{,mn}\, e^m_{\hat{k}} e^n_{\hat{\jmath}}\right)_O$. The position of the observer changes with time, therefore this quantity is a function of T.

We did not consider the time dependence of the potential U. One can prove that it is correct when $\dfrac{\Delta U}{U}\times\dfrac{r}{cT_c} << \xi O_2$ where $\dfrac{\Delta U}{U}$ is the relative change of the potential during the time T_c, at the distance r of the origin. This is generally the case.

In $G_{\hat{0}\hat{0}}$, the accuracy is limited to the terms of order of $\varepsilon^2\,\xi^3 O_6$. One can check that in such a case, the approximation $e^m_{\hat{k}} = \delta^m_{\hat{k}}$ is valid therefore $\hat{U}_{,\hat{k}\hat{\jmath}} \simeq (U_{,kj})_O$ and $\hat{U}_{,\hat{k}\hat{\jmath}\hat{\ell}} = (U_{,kj\ell})_O$. The same holds true for $\vec{\Omega}_0$ *i.e.* $\left(\vec{\Omega}_0\right)^{\hat{k}} \simeq \left(\vec{\Omega}_0\right)^k$ (see eq. (13)). Therefore, one can identify the space vectors $\overrightarrow{e}_{\hat{k}}$ of the tetrad and the space vectors $\overrightarrow{\partial}_k$ of the natural basis associated to the geocentric coordinates. This would not be valid with an higher accuracy where terms smaller than $\varepsilon^2\,\xi^3 O_6$ are considered.

Calculating $\vec{\Omega}_0$ one finds the usual following expression [5]

$$\vec{\Omega}_0 = \vec{\Omega}_{LT} + \vec{\Omega}_{dS} + \vec{\Omega}_{Th} \quad (13)$$

$$\left(\vec{\Omega}_{LT}\right)^{\hat{k}} \simeq \left(\frac{\vec{J}}{r^3} - \frac{3}{r^3}\left(\vec{J}\cdot\vec{n}\right)\vec{n} - \frac{\alpha_1}{4}\vec{\nabla}U\wedge\vec{w}\right)^k \quad (14)$$

$$\left(\vec{\Omega}_{dS}\right)^{\hat{k}} \simeq \left((1+\gamma)\,\vec{\nabla}U\wedge\vec{v}\right)^k \text{ and } \left(\vec{\Omega}_{Th}\right)^{\hat{k}} \simeq \left(\frac{1}{2}\vec{v}\wedge\frac{d\vec{v}}{dt}\right)^k \quad (15)$$

$\vec{n}$ is the direction of the satellite (fig. 3), $\vec{\Omega}_{LT}$ is the Lense-Thirring angular velocity, $\vec{\Omega}_{dS}$ and $\vec{\Omega}_{Th}$ are the de Sitter and the Thomas terms[12] :

Ω_{LT}	$\sim \dfrac{J_{Ⓢ}}{M^2_{Ⓢ}}\;\xi^2 O_4 \times \dfrac{c\xi}{R_{Ⓢ}}$
$\Omega_{dS}\sim\Omega_{Th}$	$\sim \xi^{3/2}O_3 \times \dfrac{c\xi}{R_{Ⓢ}}$

Table 4.

With $\xi \sim 0.9$, one finds

	$J_{\circledS}/M_{\circledS}^2$	Ω_{LT}	$\Omega_{dS} \sim \Omega_{Th}$
Earth	750	$\sim 10^{-14}$rads^{-1}	$\sim 10^{-12}$rads^{-1}
Jupiter	855	$\sim 10^{-12}$rads^{-1}	$\sim 10^{-11}$rads^{-1}

Table 5.

D. Aberration and deflection of the light

In the satellite, the experimental set-up is tied to a telescope which points towards a "fixed" star (see figure 1). We assume that the star is far enough for the parallax to be negligible. However the light rays suffer a gravitational deflection from the central body and an aberration which depends on the position and the velocity of the satellite. These effects result in an angular apparent velocity which must be compared to the Lense-Thirring effect.

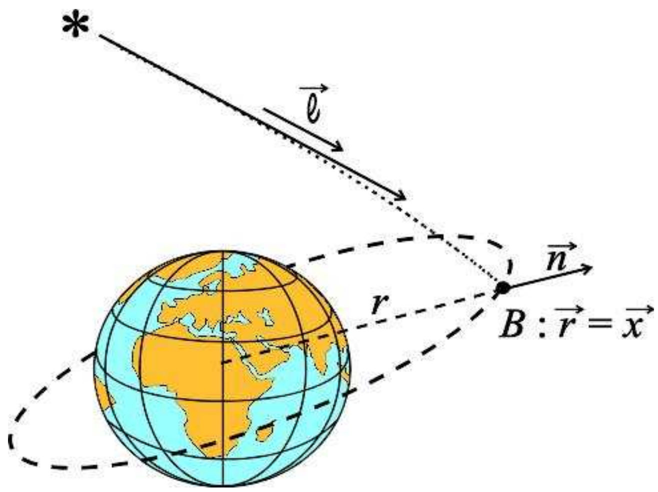

FIG. 3: The deflection of the light.

In space time, the direction of the light from the star is given by the 4-vector $L_\alpha = \left\{1, \dfrac{\partial_k \varphi}{\partial_0 \varphi}\right\}$ where φ is the phase of the light. In order to calculate the phase $\varphi(t, x^k)$ at point $\{x^k\}$ and time t we use the method which is summarized in paragraph II B. Now the line element is given by eq. (9), and ω is the angular frequency of the light at infinity. These calculations are developed in another publication [6]. Here, we just give the useful results.

The main gravitational contribution is due to the monopolar term of the Newtonian potential :

$$L_\alpha = \left\{ 1,\ -\ell^k + (1+\gamma)\,\frac{M}{r}\,\frac{n^k - \ell^k}{1 - \vec{n}\cdot\vec{\ell}} + \delta\ell^k + \delta L^k \right\} \tag{16}$$

where $\overrightarrow{\ell}$ is the unitary vector of figure 3 and $\overrightarrow{n} = \overrightarrow{r}/r$.

- The term $\delta\ell^k$ is due to the quadrupolar term of the central mass. This term is of order $J_2 O_2$ when $\overrightarrow{\ell}$ is nearly orthogonal to the plane of the orbit.
- The term due to $\dfrac{1}{2}\alpha_1 U\, w^k$, a part of g_{0k} in the metric (9), results in the modification $M_{Ⓢ} \rightarrow M = M_{Ⓢ}\left(1 - \dfrac{\alpha_1\, \vec{w}\cdot\vec{\ell}}{2\,(1+\gamma)}\right)$.
- The contribution due to the rotation of the central body is of order of $J_{Ⓢ}/r^2$. The corresponding angular velocity is of order of $J_{Ⓢ}/r^2/T \sim \dfrac{J_{Ⓢ}}{M_{Ⓢ}^2}\dfrac{\xi^{7/2}O_5}{2\pi}\dfrac{c}{R_{Ⓢ}} \sim \dfrac{\xi^{1/2}O_1}{2\pi}\Omega_{LT} << \Omega_{LT}$. It is negligible. The same conclusion holds for the term Δ in eq. (8).
- The Sun, the satellites and the other planets, give a contribution due to U_* in (8); it varies slowly with the time and it is negligible, especially within the framework of a Fourier analysis at a much higher frequency. An exception concerns the two satellites of Jupiter, Andrastea and Metis whose period is approximately 25 10^3s which is the order of the period of a satellite on a low orbit. However their mass do not exceed 10^{17}kg and the gravitational deflections remain completely negligible.

For the observer O, the space direction of the light is the four vector $\lambda^\alpha = L^\alpha - L_\beta\, u^\beta\, u^\alpha$. The components of λ^α relatively to the tetrad are $\left\{\lambda^{\hat{\alpha}}\right\} = \left(0, \overrightarrow{\lambda}\right)$. We define $\vec{\Lambda} = \Lambda\,\vec{\lambda}$ such as $-\Lambda_\alpha \Lambda^\alpha = \vec{\Lambda}\cdot\vec{\Lambda} = 1$.

The tetrad (10) is especially useful to catch the orders of magnitude of the various terms involved. However it is not the comoving tetrad that we are looking for because the telescope that points towards the far away star rotates relatively to this tetrad. The angular velocity of the telescope relatively to $\left\{e^{\alpha}_{\hat{k}}\right\}$ is $\vec{\Omega}_* = \vec{\Lambda} \wedge \dfrac{d\vec{\Lambda}}{dt}$. Straightforward calculations give

$$
\begin{aligned}
\left(\vec{\Omega}_*\right)^{\hat{k}} \;=\; & -\left(\vec{\ell} \wedge \frac{d\vec{v}}{dt}\right)^k + \left(\vec{v} \wedge \frac{d\vec{v}}{dt}\right)^k \\
& -\frac{3}{2}\left(\vec{\ell}\cdot\vec{v}\right)\left(\vec{\ell} \wedge \frac{d\vec{v}}{dt}\right)^k + \frac{1}{2}\left(\vec{\ell}\cdot\frac{d\vec{v}}{dt}\right)\left(\vec{\ell}\wedge\vec{v}\right)^k \qquad (17)\\
& -\frac{M}{r^2}\frac{1+\gamma}{1-\vec{n}\cdot\vec{\ell}}\left(\left(\vec{\ell}\wedge\vec{v}\right)^k + \left(\vec{\ell}\wedge\vec{n}\right)^k\left[\frac{\vec{\ell}\cdot\vec{v} \;-\vec{n}\cdot\vec{v}}{1-\vec{n}\cdot\vec{\ell}} - \vec{n}\cdot\vec{v}\right]\right) \\
& + \left(\vec{\ell} \wedge \frac{d\delta\vec{\ell}}{dt}\right)^k + \frac{1}{r}\times\xi^2 O_4
\end{aligned}
$$

Let us notice that we neglect the terms of order $\dfrac{1}{r}\times\xi^2 O_4$, which are much smaller than the Lense-Thirring angular velocity because $\dfrac{J_{Ⓢ}}{M^2_{Ⓢ}} >> 1$ and $\xi \sim 1$.

E. Local coordinates tied to the telescope

Now we introduce the tetrad tied to the telescope and the interferometer $u^{\alpha}_{(\sigma)}$. It is obtained from $e^{\alpha}_{\hat{\rho}}$ through a pure space rotation (*i.e.* $u^{\alpha}_{(0)} = e^{\alpha}_{\hat{0}} = u^{\alpha}$) and whose vector $u^{\alpha}_{(1)}$ points towards the far away star ($u^{\alpha}_{(1)} = -\Lambda^{\alpha}$).

At the required accuracy, it is possible to give a description of the Hyper project with the Newtonian concept of space.

The rotation of the tetrad $\left\{u^{\alpha}_{(\sigma)}\right\}$ relatively to $\left\{e^{\alpha}_{\hat{\rho}}\right\}$ is characterized by the most general angular velocity $\vec{\Omega}_{u/e} = \vec{\Omega}_* - \varpi\vec{\Lambda}$ where $-\varpi\vec{\Lambda}$ is an arbitrary angular velocity around the apparent direction of the star. The change of the tetrad $u^{\alpha}_{(\sigma)} \longleftrightarrow e^{\alpha}_{\hat{\rho}}$ is just an ordinary change of basis in the space of the observer O. In this transformation, dT, $G_{\hat{0}\hat{0}} = G_{(0)(0)}$, $G_{\hat{0}\hat{m}}dX^{\hat{m}} = G_{(0)(k)}dX^{(k)}$ and $G_{\hat{n}\hat{m}}dX^{\hat{m}}dX^{\hat{n}} = G_{(j)(k)}dX^{(j)}X^{(k)}$ behave as scalars. We obtain the local metric from eqs. (11) and (12). Then, using the expression 2 of Ψ, a straight forward calculation gives :

$$
\begin{aligned}
\Psi \;=\; & 2\,\vec{a}\cdot\vec{X} - \hat{U}_{,(k)(j)}\,X^{(k)}X^{(j)} - \frac{1}{3}\hat{U}_{,(k)(j)(\ell)}\,X^{(k)}X^{(j)}X^{(\ell)} \qquad (18)\\
& -2\sum_{(k)}\left\{\left(\vec{\Omega}_0 + \vec{\Omega}_*\right)\wedge\vec{X}\right\}^{(k)} {v_g}^{(k)} + \varepsilon^2\xi^3 O_6
\end{aligned}
$$

with $\left\{\left(\vec{\Omega}_0+\vec{\Omega}_*\right)\wedge\vec{X}\right\}\cdot\overrightarrow{v}_g = \left\{\left(\vec{\Omega}_{LT}-\left(\vec{\ell}\wedge\frac{d\vec{v}}{dt}\right)\right)\wedge\vec{X}\right\}\cdot\overrightarrow{v}_g + \eta\varepsilon\xi^{5/2}O_5$.

The Lense-Thirring contribution to Ψ is $\Psi_{LT}\sim\frac{J_{Ⓢ}}{M_{Ⓢ}^2}\xi^3\varepsilon\eta O_6$. Therefore, within the present framework, the expected accuracy is of order of

	Earth	Jupiter
$\frac{\varepsilon^2\xi^3O_6}{\left(J_{Ⓢ}/M_{Ⓢ}^2\right)\xi^3\varepsilon\eta O_6}\sim\frac{\varepsilon}{\left(J_{Ⓢ}/M_{Ⓢ}^2\right)\eta}$	18%	1.5%

Table 6.

IV. THE PHASE SHIFT

Let us assume that any quantity can be known with an accuracy $\mu\sim 10^{-4}$. This condition is not restrictive for the orbital parameters and does not seem out of the present possibilities as far as the geometry of the experimental device.

We consider that Ψ is the amount of two terms, Ψ_k and Ψ_u : the term Ψ_k is known; it can be modelled with the required accuracy while Ψ_u is unknown. The terms Ψ_k fulfills the condition $\mu\times\Psi_k\lesssim\varepsilon^2\,\xi^3O_6$. With the previous orders of magnitude one finds

		Earth	Jupiter
$\frac{\mu\hat{U}_{,(k)(j)}\;X^{(k)}X^{(j)}}{\varepsilon^2\xi^3O_6}\sim$	$\frac{\mu}{\xi O_2}\sim$	$1.6\ 10^5\in\Psi_u$	$5.7\ 10^3\in\Psi_u$
$\frac{\mu\hat{U}_{,(k)(j)(\ell)}\;X^{(k)}X^{(j)}X^{(\ell)}}{\varepsilon^2\xi^3O_6}\sim$	$\frac{\mu\varepsilon}{\xi^{1/2}O_1}\sim$	$1.5\ 10^{-2}\in\Psi_k$	$4.5\ 10^{-5}\in\Psi_k$
$\frac{\mu\left\{\left(\vec{\ell}\wedge\frac{d\vec{v}}{dt}\right)\wedge\vec{X}\right\}\cdot\overrightarrow{v}_g}{\varepsilon^2\xi^3O_6}\sim$	$\frac{\mu\eta}{\varepsilon\xi O_2}\sim$	$1.2\ 10^3\in\Psi_u$	$4.5\ 10^2\in\Psi_u$
$\frac{\mu\eta\varepsilon\xi^{5/2}O_5}{\varepsilon^2\xi^3O_6}\sim$	$\frac{\mu\eta}{\varepsilon\xi^{1/2}O_1}\sim$	$3\ 10^{-2}\in\Psi_k$	$6\ 10^{-2}\in\Psi_k$

Table 7.

Ψ_u reads

$$\Psi_u = 2\,\vec{a}\cdot\vec{X}-\hat{U}_{,(k)(j)}\;X^{(k)}X^{(j)}-2\left\{\vec{\Omega}\wedge\vec{X}\right\}\cdot\overrightarrow{v}_g+\varepsilon^2\,\xi^3O_6/\mu \quad (19)$$

where the contribution $\hat{U}_{,(k)(j)}\;X^{(k)}X^{(j)}$ needs to be defined with an accuracy better than $\varepsilon^2\,\xi^3O_6/\mu$. This implies that any known perturbation δU

can be included in Ψ_k when $\dfrac{\delta U}{U}$ does not exceed the value given in table 8 below :

	Earth	Jupiter
$\dfrac{\delta U}{U} \lesssim \dfrac{\varepsilon\xi O_2}{\mu}$	$2\ 10^{-8}$	10^{-8}

Table 8.

The quadrupolar contribution is bigger than 10^{-3}, it cannot be considered in known term, however higher multipole can be included in Ψ_k if the accuracy μ is smaller than 10^{-6} for the Earth instead of 10^{-4} and 10^{-9} for Jupiter. The accuracy $\mu \sim 10^{-6}$ remains a very difficult challenge as far as the geometry of the set-up is concerned (A. Landragin, private communication).

With the same order of magnitude for μ, we obtain

$$\left(\vec{\Omega}\right)^{(k)} = \left(\vec{\Omega}_{LT} - \varpi\overrightarrow{\Lambda} - \overrightarrow{\Lambda} \wedge \frac{d\overrightarrow{v}}{dt}\right)^{(k)} \tag{20}$$

where $\vec{\Omega}_{LT}$ is deduced from (14) and $\dfrac{d\overrightarrow{v}}{dt} \simeq \overrightarrow{a} - \overrightarrow{\nabla}U \simeq \overrightarrow{a} + \dfrac{M_{\circledS}}{r^2}\overrightarrow{n}$.

The satellites, such as the Moon for the Earth, can bring a contribution to $\hat{U}_{,(k)(j)}\ X^{(k)}X^{(j)}$ at the required level of accuracy but with such a value of μ, it could be included in Ψ_k.

Of course when modelling Ψ_k, one must be sure that every quantity is known at the required accuracy. This must hold for any value of μ, this is a necessary condition. Therfore the following relation must hold true : $\dfrac{\Delta U}{U} \lesssim \varepsilon\xi O_2$

	Earth	Jupiter
$\dfrac{\Delta U}{U} \lesssim \varepsilon\xi O_2$	$2\ 10^{-12}$	10^{-12}

Table 9.

Such an accuracy is not achieved for Jupiter. For the Earth, considering the difference between the various models (Godard Earth model 9 and 10) it appears that the values of the high order multipoles are neither known nor consistent at the required level (10^{-12}). One can hope that the lack of precision on the J_{kn} coefficients [13] is not important for $n \neq 0$ because the diurnal rotation modulates the frequency of the corresponding contribution in $\hat{U}_{,(k)(j)}\ X^{(k)}X^{(j)}$. However it is necessary to increase our knowledge of the axisymmetrical potential of the central body in order that $\xi^k\ \Delta J_{k0} \lesssim \varepsilon\xi O_2$ where ΔJ_{k0} is the uncertainty on $J_{k0} = J_k$. Such a relation holds true for $k = 2$. It could be presently achieved with low values of ξ (on high orbits) but the Lense-Thirring effect is proportional to ξ^3 (see table 4 above) and it seems impossible to measure the Lense-Thirring effect for $\xi << 1$ in not to far a future.

This question is crucial but Hyper might, itself, bring an answer to this question by the means of the time analysis. Now we forget these problems because in the simple case that we consider the quadratic quantities do not bring any contribution to the signal.

A. Significant terms in Ψ_u

We consider that the motion of the satellite takes place in the (x, y)-plane while the vector $\overrightarrow{\ell}$ lies in the (x, z)-plane. We assume that the eccentricity, e, does not exceed $\xi^{1/2}O_1$.

We define

$$\vec{J} = J_x\,\vec{e}_{\hat{1}} + J_y\,\vec{e}_{\hat{2}} + J_z\,\vec{e}_{\hat{3}}\ , \quad \vec{n} = \cos\theta\,\vec{e}_{\hat{1}} + \sin\theta\,\vec{e}_{\hat{2}} \tag{21}$$

$$-\vec{\ell} = \cos\alpha\,\vec{e}_{\hat{1}} + \sin\alpha\,\vec{e}_{\hat{3}}\ , \quad \vec{w} = w_x\,\vec{e}_{\hat{1}} + w_y\,\vec{e}_{\hat{2}} + w_z\,\vec{e}_{\hat{3}} \tag{22}$$

$\vec{J}$, $\vec{\ell}$ and $\vec{w}$ are constant vectors. The angle θ and the distance r depend on the time T.

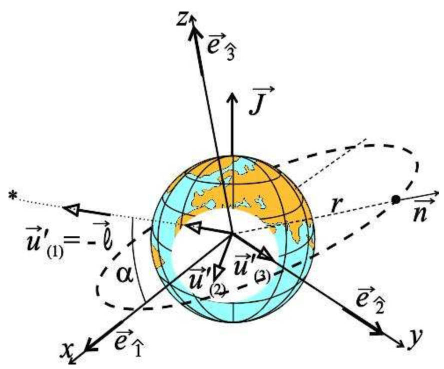

FIG. 4: The satellite and the fixed star

First we define the spatial triad $\vec{u}'_{(n)}$:

$$\vec{u}'_{(1)} = \cos\alpha\,\overrightarrow{e}_{\hat{1}} + \sin\alpha\,\overrightarrow{e}_{\hat{3}}, \quad \vec{u}'_{(2)} = \sin\alpha\,\overrightarrow{e}_{\hat{1}} - \cos\alpha\,\overrightarrow{e}_{\hat{3}}, \quad \vec{u}'_{(3)} = \overrightarrow{e}_{\hat{2}} \tag{23}$$

Let us outline that we have defined $\left(\vec{u}'_{(1)}\right)^{\hat{k}} = -\left(\vec{\Lambda}\right)^{\hat{k}} + \xi^{1/2}O_1$. Then, in order to obtain the final tetrad $u^{\alpha}_{(\sigma)}$, we perform an arbitrary rotation around $\vec{\Lambda}$:

$$\begin{aligned} \vec{u}_{(1)} &= -\vec{\Lambda} = \vec{u}'_{(1)} + \xi^{1/2}O_1 \\ \vec{u}_{(2)} &= \vec{u}'_{(2)}\cos\sigma + \vec{u}'_{(3)}\sin\sigma + \xi^{1/2}O_1 \\ \vec{u}_{(3)} &= -\vec{u}'_{(2)}\sin\sigma + \vec{u}'_{(3)}\cos\sigma + \xi^{1/2}O_1 \end{aligned} \tag{24}$$

where $-\dfrac{d\sigma}{dT} = -\varpi$ is the angular velocity of the triad $\left\{\vec{u}_{(k)}\right\}$ relatively to $\left\{\vec{u}'_{(k)}\right\}$.

We can now assume that the experimental set-up is comoving with the triad $\vec{u}_{(n)}$ whose vector $\vec{u}_{(1)}$ points towards the fixed star.

During the flight of the atom, the quantity $\hat{U}_{,(k)(j)}$ in equation 19 does not remain constant because the position of the satellite changes. One can consider that the coordinate, $X = X^{(1)}$ of the atom is a function of the time: $X = v_g\left(T - T_0\right)$.

Therefore we expand $\hat{U}_{,(k)(j)} = \hat{U}_{,(k)(j)}\left(T_0\right) + \hat{U}_{,(k)(j)(\ell)}\left(T_0\right)\ v^{(\ell)} \times \dfrac{X}{v_g}$ where $v^{(\ell)}\overrightarrow{u}_{(\ell)}$ is the orbital velocity.

Before performing explicit calculation we notice that $\hat{U}_{,(k)(j)}\left(T_0\right) X^{(k)}X^{(j)}$ will not bring any contribution to the phase difference 3 because O_S and O'_S are two centers of symmetry. For $\mu < 10^{-6}$ the term $\hat{U}_{,(k)(j)(\ell)}\left(T_0\right)\ \dfrac{v^{(\ell)}}{v_g} \times X\ X^{(k)}X^{(j)} \sim \varepsilon^3\xi^{5/2}O_5/\eta$ can be included in Ψ_k for Jupiter ($\dfrac{\mu\varepsilon^3\xi^{5/2}O_5/\eta}{\varepsilon^2\xi^3O_6} < 1$), but it must considered for the Earth $\left(\dfrac{\mu\varepsilon^3\xi^{5/2}O_5/\eta}{\varepsilon^2\xi^3O_6} \sim 5 > 1\right)$. However the quadrupole does not bring any contribution to the phase difference that we calculate from Ψ_u.

In equation 19, the spin $\varpi\vec{u}_{(1)}$ does not bring any contribution in the term $\left\{\vec{\Omega} \wedge \vec{X}\right\} \cdot \overrightarrow{v}_g$ because $\vec{u}_{(1)}$, $\overrightarrow{X}$ and $\vec{v}_g$ are in the same plane.

Then, one obtains

$$\begin{array}{rl|l} \Psi_u = & -2\left(\left(\left(\dfrac{\vec{J}}{r^3} - \dfrac{3\left(\vec{J}\cdot\vec{n}\right)}{r^3}\vec{n} + \dfrac{\alpha_1\ M_{ⓢ}}{4\,r^2}\,\vec{n}\wedge\vec{w}\right)\wedge\overrightarrow{X}\right)\cdot\vec{v}_g\right. & A \\ & -2\dfrac{M_{ⓢ}}{r^2}\left(\left(\vec{u}_{(1)}\wedge\vec{n}\right)\wedge\overrightarrow{X}\right)\cdot\vec{v}_g + 2\left(\left(\vec{u}_{(1)}\wedge\vec{a}\right)\wedge\overrightarrow{X}\right)\cdot\vec{v}_g & B \\ & -\dfrac{6M_{ⓢ}}{r^4}\left(\overrightarrow{X}\cdot\dfrac{\overrightarrow{v}}{v_g}\right)\left(\overrightarrow{X}\cdot\overrightarrow{n}\right)X & C \\ & +2\,\vec{a}\cdot\overrightarrow{X} & D \end{array} \qquad (25)$$

where $\overrightarrow{n} = \overrightarrow{n}\,(T_0)$.

In expression (eq. (25)) of Ψ_u one can assume that $r = r_0$ is a constant because we assume that the excentricity is small $e \lesssim O_1$, therefore the corrections are included in Ψ_k.

Moreover, for the same reason, one can drop the terms of order O_1 in the expression of the tetrads. Therefore, it is clear that we can consider the space as the ordinary space of Newtonian physics and that the usual formulae to change the basis $\overrightarrow{\partial}_k$ into $\overrightarrow{e}_{\hat{k}}$ or $\overrightarrow{u}_{(k)}$are valid.

In (25), the terms of lines A and B are due to various rotations : respectively the Lense-Thirring rotation and the aberration. The term of line C is due to the displacement of the satellite during the flight time of the atom and the term of line D corresponds to some residual acceleration due to the fact that point O is not exactly in free fall.

B. The phase differences

We use the expression (25) of Ψ_u in order to calculate $\delta\varphi$ given by 3. We find

$$\begin{aligned} \delta\varphi \;=\; & -2\frac{mc}{\hbar\,r}S\left(\frac{\vec{J}}{r_0^2} - \frac{3\left(\vec{J}\cdot\vec{n}\right)}{r_0^2}\vec{n} + \frac{\alpha_1\ M_{ⓢ}}{4\,r_0}\,\vec{n}\wedge\vec{w}\right)\cdot\vec{u}_{(2)} \\ & -2\frac{mc}{\hbar r}S\,\frac{M_{ⓢ}}{r_0}\,\left(\vec{u}_{(1)}\wedge\vec{n}\right)\cdot\vec{u}_{(2)} - 2\frac{mc}{\hbar}S\left(\vec{u}_{(1)}\wedge\frac{\overrightarrow{a}}{c^2}\right)\cdot\vec{u}_{(2)} \qquad (26) \\ & -\frac{4\pi\left(cT_D\right)^2}{\lambda}\left(\vec{u}_{(3)}\cdot\frac{\vec{a}_{O_S}}{c^2}\right) \\ & -\frac{mc}{2\,\hbar r}S\,\frac{\left(cT_D\right)^2}{r_0^2}\frac{M_{ⓢ}}{r}\,\left(\left(\vec{u}_{(1)}\cdot\vec{n}\right)\left(\vec{u}_{(3)}\cdot\frac{\overrightarrow{v}}{c}\right) + \left(\vec{u}_{(3)}\cdot\vec{n}\right)\left(\vec{u}_{(1)}\cdot\frac{\overrightarrow{v}}{c}\right)\right) \end{aligned}$$

where $S=\dfrac{4\pi\hbar}{\lambda m}v_gT_D^2$ is the area of the Sagnac loop. As we mentionned before, $\vec{J}$ and $M_{\circledS}$ are expressed in geometrical units.

The two interferometers of the same ASU are assumed to lie in the same plane but not necessarily with their center of symmetry O_S and O'_S at the same point. Therefore adding and subtracting the phase differences delivered by the two interferometers one finds the two basic quantities which are measured by the set-up *i.e.* : $\mu_1=\dfrac{1}{2}\left(\delta\varphi'-\delta\varphi\right)$ and $\mu_2=\dfrac{1}{2}\left(\delta\varphi'+\delta\varphi\right)$.

We define the shift $\overrightarrow{\delta}=\overrightarrow{X}_{O'_S}-\overrightarrow{X}_{O_S}$ and the acceleration $\overrightarrow{a}=\dfrac{1}{2}\left(\vec{a}_{O_S}+\vec{a}_{O'_S}\right)$ where $\vec{a}_{O_S}$ and $\vec{a}_{O'_S}$ are the accelerations at point O_S and O'_S. We drop several terms which can be included into Ψ_k. Then we obtain the quantities which can be measured :

$$
\begin{aligned}
\mu_1+\frac{2v_g}{c}\mu_2 \;=\;& \frac{8\pi}{\lambda}\left(cT_D\right)^2\left\{\frac{M_{\circledS}}{r_0^2}\vec{u}_{(3)}\cdot\vec{n}+\vec{\Omega}_{LT}\cdot\vec{u}_{(2)}\right\}\frac{v_g}{c} \\
&-\frac{2\pi\left(cT_D\right)^2}{\lambda r_0}\frac{M_{\circledS}}{r_0}\left(\vec{u}_{(3)}\cdot\frac{\overrightarrow{\delta}}{r}-3\left(\vec{n}\cdot\vec{u}_{(3)}\right)\left(\vec{n}\cdot\frac{\overrightarrow{\delta}}{r}\right)\right) \\
&-\frac{2\pi}{\lambda}\left(cT_D\right)^2\left(\frac{v_gT_D}{r_0}\right)^2\frac{M_{\circledS}}{r_0^2}\left\{\left(\vec{u}_{(1)}\cdot\vec{n}\right)\left(\vec{u}_{(3)}\cdot\frac{\overrightarrow{v}}{c}\right)\right. \\
&\left.+\left(\vec{u}_{(3)}\cdot\vec{n}\right)\left(\vec{u}_{(1)}\cdot\frac{\overrightarrow{v}}{c}\right)\right\}
\end{aligned}
\tag{27}
$$

$$
\mu_2 \;=\; \frac{1}{2}\left(\delta\varphi'+\delta\varphi\right)=-\frac{4\pi\left(cT_D\right)^2}{\lambda}\left\{\frac{\overrightarrow{a}}{c^2}\cdot\vec{u}_{(3)}\right\}
\tag{28}
$$

C. Discussion

We define α as the direction of the fixed star (fig. (4)), and the projection, $\overrightarrow{J}_{\shortparallel}$ of $\overrightarrow{J}$ on the plane of the orbit :

$\overrightarrow{J}_{\shortparallel}=J_{\shortparallel}\left(\cos\theta_J\ \overrightarrow{e}_{\hat{1}}+\sin\theta_J\ \overrightarrow{e}_{\hat{2}}\right)$ and $\overrightarrow{w}_{\shortparallel}=w_{\shortparallel}\left(\cos\theta_w\ \overrightarrow{e}_{\hat{1}}+\sin\theta_w\ \overrightarrow{e}_{\hat{2}}\right)$.
Then

$$
\begin{aligned}
\mu_1+2\frac{v_g}{c}\mu_2 \;=\;& \frac{2\pi\left(cT_D\right)^2}{\lambda r_0}\times\left\{K_0+K_\sigma+K_{2\sigma}+K_{2\theta}\right. \\
&\left.+K_{\theta-\sigma}+K_{\theta+\sigma}+K_{2\theta-\sigma}+K_{2\theta+\sigma}+K_{2\theta-2\sigma}+K_{2\theta+2\sigma}\right\}
\end{aligned}
\tag{29}
$$

with

$$K_0 = \frac{M_{\circledS}}{4\,r_0}\left(3\sin^2\alpha - 1\right) \times \frac{\delta^{(3)}}{r_0} \tag{30}$$

$$\begin{aligned} K_\sigma &= \frac{v_g}{c}\left\{\left[(1-\sin\alpha)\cos(\sigma+\theta_J) - (1+\sin\alpha)\cos(\sigma-\theta_J)\right] \times \frac{J_{\shortparallel}}{r_0^2}\right. \\ &\quad \left. -4\cos\alpha\,\cos\sigma \times \frac{J^{\hat{3}}}{r_0^2}\right\} - \frac{3M_{\circledS}}{2r_0}\cos\alpha\sin\alpha\sin\sigma \times \frac{\delta^{(1)}}{r_0} \end{aligned} \tag{31}$$

$$K_{2\sigma} = \frac{3M_{\circledS}}{4r_0}\left(1-\sin^2\alpha\right)\left[\sin(2\sigma) \times \frac{\delta^{(2)}}{r_0} + \cos(2\sigma) \times \frac{\delta^{(3)}}{r_0}\right] \tag{32}$$

$$K_{2\theta} = -\frac{3M_{\circledS}}{4r_0}\left(1-\sin^2\alpha\right)\cos(2\theta) \times \frac{\delta^{(3)}}{r_0} \tag{33}$$

$$\begin{aligned} K_{\theta-\sigma} &= \frac{M_{\circledS}}{r_0}\frac{v_g}{c} \times \left\{-2\,(1+\sin\alpha)\,\sin(\theta-\sigma)\right. \\ &\quad + \frac{\alpha_1}{2}\cos\alpha\,\sin(\theta-\sigma-\theta_w) \times \frac{w_{\shortparallel}}{c} \\ &\quad \left. + \frac{\alpha_1}{2}(1+\sin\alpha)\sin(\theta-\sigma) \times \frac{w^{\hat{3}}}{c}\right\} \end{aligned} \tag{34}$$

$$\begin{aligned} K_{\theta+\sigma} &= \frac{M_{\circledS}}{r_0}\frac{v_g}{c} \times \left\{-2\,(1-\sin\alpha)\,\sin(\theta+\sigma)\right. \\ &\quad + \frac{\alpha_1}{2}\cos\alpha\,\sin(\theta+\sigma-\theta_w) \times \frac{w_{\shortparallel}}{c} \\ &\quad \left. - \frac{\alpha_1}{2}(1-\sin\alpha)\sin(\theta+\sigma) \times \frac{w^{\hat{3}}}{c}\right\} \end{aligned} \tag{35}$$

$$\begin{aligned} K_{2\theta-\sigma} &= -3\frac{v_g}{c}(1+\sin\alpha)\cos(2\theta-\sigma-\theta_J) \times \frac{J_{\shortparallel}}{r_0^2} \\ &\quad + \frac{3M_{\circledS}}{4r_0}\cos\alpha\,(1+\sin\alpha)\sin(2\theta-\sigma) \times \frac{\delta^{(1)}}{r_0} \\ &\quad - \left(\frac{M_{\circledS}}{r_0}\right)^{3/2}\frac{cv_gT_D^2}{2\,r_0^2}(1+\sin\alpha)\cos(2\theta-\sigma) \end{aligned} \tag{36}$$

$$\begin{aligned} K_{2\theta+\sigma} &= 3\frac{v_g}{c}(1-\sin\alpha)\cos(2\theta+\sigma-\theta_J) \times \frac{J_{\shortparallel}}{r_0^2} \\ &\quad + \frac{3M_{\circledS}}{4r_0}\cos\alpha\,(1-\sin\alpha)\sin(2\theta+\sigma) \times \frac{\delta^{(1)}}{r_0} \\ &\quad - \left(\frac{M_{\circledS}}{r_0}\right)^{3/2}\frac{cv_gT_D^2}{2\,r_0^2}(1-\sin\alpha)\cos(2\theta+\sigma) \end{aligned} \tag{37}$$

$$K_{2\theta-2\sigma} = \frac{3M_{Ⓢ}}{8\,r_0}\,(1+\sin\alpha)^2 \times \left\{\sin(2\theta-2\sigma)\times\frac{\delta^{(2)}}{r_0} - \cos(2\theta-2\sigma)\times\frac{\delta^{(3)}}{r_0}\right\} \tag{38}$$

$$K_{2\theta+2\sigma} = -\frac{3M_{Ⓢ}}{8\,r_0}\,(1-\sin\alpha)^2 \times \left\{\sin(2\theta+2\sigma)\times\frac{\delta^{(2)}}{r_0} + \cos(2\theta+2\sigma)\times\frac{\delta^{(3)}}{r_0}\right\} \tag{39}$$

Each of these terms, except K_0, has a specific frequency. They can be measured and distinguished from each other.

The Lense-Thirring effect due to the angular momentum of the central body appears in the terms K_σ and $K_{2\theta\pm\sigma}$ while the possible existence of a preferred frame appears in $K_{\theta\pm\sigma}$ which depends on the components of $\alpha_1\,\overrightarrow{w}$.

The signal due to the Lense-Thirring effect is associated with the signal due to $\delta^{(1)}$. Today, it seems impossible to reduce $\delta^{(1)}$ significantly, this is the reason why it should be calculated from the Fourier analysis of the signal itself altogether with the velocity $\alpha_1\,\overrightarrow{w}$.

The interest of the spin is obvious. If σ is constant (no spin) the signal is the sum of two periodic signals with frequency ν_O and $2\nu_O$ where ν_O is the orbital frequency of the satellite ; therefore one ASU gives two informations (two functions of the time). When the satellite spins, we get 9 functions of the time t. The information is much more important in this case.

V. CONCLUSION

In Hyper, the Lense-Thirring effect is associated with many perturbations which cannot be cancelled. We have exhibited the various terms that one needs to calculate in order to obtain the full signal and we have emphasized the necessity to increase our knowledge of the Newtonian gravitational potential. This is still more crucial for Jupiter despite the fact that the Lense-Thirring effect is much bigger.

Using four parameters, ξ, ε, η and μ defined in table 2, we have also sketched a method to take into account the residual gravitational field in a nearly free falling satellite, namely the tidal and higher order effects.

Compared with GPB, the principle of the measure is not the same, the difficulties are quite different but the job is not easier. For instance, considering the quantities K_σ or $K_{2\theta\pm\sigma}$ above, one can check that, for an Earth satellite, $\delta^{(1)}$ must remain smaller than 2nm for the corresponding signal to remain smaller than the Lense-Thirring one. It does not seem that such a precision can be controlled in the construction of the experimental device itself. It is therefore necessary to measure $\delta^{(1)}$ with such an accuracy.

What can be deduced from the time analysis depends on the accuracy of the various parameters. From $K_{2\theta\pm2\sigma}$ we deduce α. Then from $K_{\theta\pm\sigma}$ we obtain $\alpha_1 w_{\shortparallel}/c$ and $\alpha_1 w^{\hat{3}}/c$ as two functions of θ_w. Therefore one can check if $\alpha_1 w = 0$ or not.

From $K_{2\theta\pm\sigma}$ one can calculate $\dfrac{M_{\circledS} c}{r_0^2 v_g}\delta^{(1)}$ as a function $G_{(+)}$ of $J_{\shortparallel}/r_0^2$, θ_J and $\left(\dfrac{M_{\circledS}}{r_0}\right)^{3/2} \dfrac{c^2 T_D^2}{r_0^2}$ and as a different function, $G_{(-)}$ of the same arguments. One could check the equality $G_{(+)} = G_{(-)}$. If we assume that $\overrightarrow{J}_{\circledS}$ is known, then θ_J is known and from the equality $G_{(+)} = G_{(-)}$ we deduce the value of $J_{\shortparallel}$. Using the relation $J_{\shortparallel} = \dfrac{1+\gamma+\alpha_1/4}{2}\left(\overrightarrow{J}_{\circledS}\right)_{\shortparallel}$ one could check whether $\gamma + \alpha_1/4 = 1$.

K_σ would give $\dfrac{M_{\circledS} c}{r_0^2 v_g}\delta^{(1)}$ as a function of $J_{\shortparallel}/r_0^2$, θ_J and $J^{\hat{3}}/r_0^2$. Using the previous results, we obtain $J^{\hat{3}}$. The relation $J^{\hat{3}} = \dfrac{1+\gamma+\alpha_1/4}{2}\left(\overrightarrow{J}_{\circledS}\right)^{\hat{3}}$ gives an other test of the value of $\gamma + \alpha_1/4$.

But over all, the best test would be that the signal (as a function of the time) fits the theoretical prediction.

As a final conclusion let us put forwards that the geometric scheme which has been used is just a preliminary contribution to the discussion on the feasibility of Hyper. Only a more powerful model can answer the question. This model should take into account all the gravitational perturbations that we have outlined here and it should consider the interaction between laser fields and matter waves in more a realistic manner. Such an approach has been recently developed [7], [8] it could give definitive results in the future.

[1] R. Bingham *et al.*, *HYPER Hyper-Precision Cold Atom Interferometry in Space Assessment Study Report*, vol. 10 (ESA-SCI, 2000).
[2] B. Linet and P. Tourrenc, Can. J. Phys. **54**, 1129 (1976).
[3] W.-Q. Li and W.-T. Ni, J. Math. Phys. **20**, 1473 (1979).
[4] C. Will, *Theory and experiment in gravitational physics* (Cambridge University Press, 1981).
[5] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (San Francisco: W.H. Freeman and Co., 1973, 1973).
[6] M.-C. Angonin, X. Ovido, and P. Tourrenc, GRG **36**, 411 (2004).
[7] C. Antoine and C. Bordé, J. Opt. B **5**, S199 (2003).
[8] C. Bordé, accepted to GRG (2004).
[9] The Lense-Thirring effect results also in a secular precession which is not considered here but in Gravitational Probe B : a NASA experiment which

is planed to be launched on the 6th of December 2003.
[10] An arbitrary constant can always be added to U_*. It is chosen in such a way that zero is the mean value of U_* at point O in the satellite.
[11] Notice that the quantity $\xi^{1/2}O_1$ is what is called O_1 in Will's book quoted above
[12] The Thomas term reads $\vec{\Omega}_{Th} = \frac{1}{2}\vec{v} \wedge \vec{A}$ where $\vec{A}$ is the "acceleration". From the relativistic point of view, it would be better to define the Thomas term with the local physical acceleration, $\vec{A} \simeq \frac{d\vec{v}}{dt} - \vec{\nabla}U$, rather than the acceleration, $\frac{d\vec{v}}{dt}$, relatively to the geocentric frame.
[13] notations of the paper by Ch. Marchal in Bulletin du museum d'histoire naturelle, 4ème série, section C **18,** 517 (1996)